Journal of Statistical Physics, Vol. 63, Nos. 1/2, 1991

A New Rigorous Upper Bound for the Inverse Critical Temperature of the Two-Dimensional Coulomb Gas

Gastão de Almeida Braga[1]

Received August 7, 1990; final October 26, 1990

In this paper we show how to improve the recent result $\beta_c \leqslant 17.2\pi$ on the inverse critical temperature for the two-dimensional Coulomb gas at low density to get the following upper bound: $\beta_c \leqslant 16\pi$.

KEY WORDS: Two-dimensional Coulomb gas, Kosterlitz–Thouless phase transition, inverse critical temperature; dipole phase; plasma phase; multiscaling expansion.

In a recent paper,[1] it has been shown how the energy estimate for the two-dimensional lattice Coulomb gas at low density could be improved to get the upper bound

$$\beta_c \leqslant 17.2\pi$$

where $\beta_c = 1/kT_c$ and T_c is the critical temperature.

In this short communication, we show how to improve this bound to get

$$\beta_c \leqslant 16\pi \tag{1}$$

We start by reviewing the argument in ref. 1: let ρ_{k+1} and σ_{k+1} be distinct neutral charge densities living in the background of charge distributions being renormalized by complex translations. ρ_{k+1} and σ_{k+1} are functions of $j \in Z^2$. They satisfy the following conditions:

[1] Universidade Federal de Minas Gerais, Departamento de Matemática-ICEX, Caixa Postal 1621, C.E.P. 31270, Belo Horizonte, M. G., Brazil.

1. The perimeter of $\mathrm{supp}(\rho_{k+1})$ is approximately l_k, where supp is the support of $\rho_{k+1}(j)$.
2. $\mathrm{supp}(\rho_{k+1}) \subset B_{k+1}$, where B_{k+1} is an $l_{k+1} \times l_{k+1}$ square.
3. $\mathrm{dist}(\rho_{k+1}, \sigma_{k+1}) \geq l_{k+2}$, where $\mathrm{dist}(\cdot, \cdot)$ stands for the distance between the supports of charge densities.

l_k is the length at scale k (see ref. 2 for details). The boundary contribution to the "energy estimate" is finite (and small in the low-density regime) if the following inequality holds:

$$l_k \times \left(\frac{l_{k+1}}{l_{k+2}}\right)^2 = l_k^{1+2\theta-2\theta^2} < \delta^k \tag{2}$$

where $0 < \delta < 1$ and θ is the exponent characterizing the rate of growth of the sequence $\{l_k\}$, $l_{k+1} = l_k^\theta$. Inequality (2) is satisfied for $\theta > (\sqrt{3}+1)/2$, which gives the result $\beta_e \leq 17.2\pi$.

One may ask what happens if neutral charge densities $\rho_{k+1} \subset B_{k+1}$, whose perimeter is of order l_k, are further away from each other. For such, we generalize condition 3 given above:

$$\mathrm{dist}(\rho_{k+1}, \sigma_{k+1}) \geq l_{k+1+n} \tag{3}$$

where $n = 1, 2, 3,\ldots$. The distance requirement (3) can be obtained by imposing that dipoles formed at scale k will stay renormalized at scale $k+n$, as the next proposition shows. We observe that it is an unnatural condition and it is the reason why we cannot obtain a result better than (1). In what follows, $z(\rho_k)$ stands for the activity of ρ_k. We remark that (see ref. 2 for details)

$$\begin{aligned} z_{k+1} &= z_k^{1+\alpha\varepsilon} \\ \frac{l_{k+1}}{l_k} &= z_k^{-\alpha} \geq l_k^{\theta-1} \end{aligned} \tag{4}$$

where z_k is the activity at scale k; $z_0 = z$ is the initial activity; ε is a positive number to be chosen later; α is the exponent relating the length scales and activities. Inequality (4) is satisfied once

$$\alpha\varepsilon > \theta - 1 \tag{5}$$

Proposition 1. Let ρ_{k+1} be a neutral charge density localized on B_{k+1} such that

$$\rho_{k+1} = \sum c_i \rho_k^i, \qquad \text{where } i \geq 2$$

$$|z(\rho_k)| < z_k$$

Suppose that, during the induction $k+2 \to k+3 \to \cdots k+n$, ρ_k remains isolated (i.e., $\rho_k = \rho_{k+n}$). If

$$\frac{\varepsilon}{\varepsilon+2} > \frac{2\alpha\varepsilon}{\varepsilon+2} + [(1+\alpha\varepsilon)^n - 1] \tag{6}$$

then $|z(\rho_{k+n})| < z_{k+n}$.

Proof. For each scale change $L_{k+i} \to L_{k+i+1}$ we get an entropy factor of $(L_{k+i+1}/L_{k+i})^2$ for $z(\rho_{k+n})$. Then

$$\begin{aligned} |z(\rho_{k+n})| &< |z(\rho_k)|^2 \, (l_{k+1}/l_k)^4 \, (l_{k+2}/l_{k+1})^2 \cdots (l_{k+n}/l_{k+n-1})^2 \\ &< z_k^2 (l_{k+1}/l_k)^4 \, (l_{k+2}/l_{k+1})^2 \cdots (l_{k+n}/l_{k+n-1})^2 \end{aligned} \tag{7}$$

Imposing the condition

$$|z(\rho_{k+n})| < z_{k+n} \tag{8}$$

we rewrite the product (7) and z_{k+n} in terms of z_k and compare exponents to conclude that (6) is a sufficient condition for (8) to be true. ▮

The parameter ε is chosen such that

$$0 < \varepsilon < \frac{\beta}{4\pi(1+S)} - 2 \tag{9}$$

S stands for the boundary contributions coming from the "energy estimate." S is a function of z, the initial activity, and it goes to zero as z goes to zero (see ref. 2 for details). The equivalent to (2) will be

$$l_k \times \left(\frac{l_{k+1}}{l_{k+1+n}} \right)^2 = l_k^{1+2\theta-2\theta^{n+1}} < \delta^k$$

which holds for values of θ satisfying the inequality

$$1 + 2\theta - 2\theta^{n+1} < 0 \tag{10}$$

In the Appendix we show that α and θ can be found satisfying (5), (6), and (10) if and only if ε satisfies the lower bound

$$\varepsilon > \frac{4\theta^2 - 4\theta + 2}{2\theta - 1}$$

which implies, after using (9),

$$\beta > 8\pi(1+S) \times \frac{2\theta^2}{2\theta-1} > 8\pi(1+S) \times \frac{2\theta_c^2}{2\theta_c - 1} \tag{11}$$

where $\theta_c = \theta_c(n)$ is the root of (13). Taking the limits $\theta_c(n) \searrow 1$ as $n \to \infty$ (see the Appendix) and $S \to 0$ as $z \to 0$, we get

$$\beta > 16\pi$$

and therefore the claimed result (1).

We observe that, for $n = 1$, our result (11) is the same as the one appearing in ref. 1. To see it, substitute (13) in (11) to obtain

$$\beta > 8\pi \times \frac{\theta_c}{2 - \theta_c^n}$$

which, for $n = 1$, gives

$$\beta > 8\pi \times \frac{\theta_c}{2 - \theta_c} \tag{12}$$

which is the formula for β_c appearing in ref. 1.

APPENDIX

Let $\theta_c(n)$ be the real positive root closest to 1 of

$$\frac{1 + 2\theta}{2\theta} = \theta^n \tag{A1}$$

From the intersection between the graphics of θ^n and $1 + 1/2\theta$ it follows that

$$\theta_c(n) > 1$$

$$\theta_c(n+1) < \theta_c(n)$$

$$\theta_c(n) \searrow 1 \qquad \text{as } n \to \infty$$

Note that the inequality (10) is satisfied by any $\theta > \theta_c(n)$ because $2(\theta^{n+1} - \theta) > 2\theta_c(\theta_c^n(n) - 1) = 1$.

α_1 is defined as the root of

$$\varepsilon/(\varepsilon + 2) = 2\alpha\varepsilon/(\varepsilon + 2) + [(1 + \alpha\varepsilon)^n - 1] \equiv f(\alpha)$$

Observe that $f(\alpha)$ is an increasing polynomial function of α. Let $\alpha_e(n) \equiv [\theta_c(n) - 1]/\varepsilon$. Observe that, for $\theta > \theta_c(n)$, the following inequality is true:

$$\frac{\theta - 1}{\varepsilon} = \alpha > \alpha_c = \frac{\theta_c - 1}{\varepsilon}$$

Proposition 2. $f(\alpha_c) < f(\alpha_1)$ if and only if

$$\varepsilon > (4\theta_c^2 - 4\theta_c + 2)/(2\theta_c - 1)$$

Proof.

$$f(\alpha_c) = \frac{2(\theta_c - 1)}{\varepsilon + 2} + \theta_c^n - 1 = \frac{2(\theta_c - 1)}{\varepsilon + 2} + \frac{1}{2\theta_c} = \frac{2(\theta_c - 1) + \varepsilon/2\theta_c + 1/\theta_c}{\varepsilon + 2}$$

Therefore, we have

$$f(\alpha_c) < \varepsilon/(\varepsilon + 2) \Leftrightarrow 2(\theta_c - 1) + \frac{\varepsilon}{2\theta_c} + \frac{1}{\theta_c} < \varepsilon$$

from which the result follows. ∎

Therefore, by the continuity of $f(\alpha)$, we can choose $\theta > \theta_c$, close enough to θ_c, such that the inequalities (5), (6), and (10) are satisfied.

Remark. After the completion of this work the author received a preprint[3] in which the Kosterlitz–Thouless phase is established for $\beta > 8\pi$ and small z.

REFERENCES

1. D. H. Marchetti, Comments on "Power law fall off in the two dimensional lattice Coulomb gas," Rutgers University preprint, *J. Stat. Phys.*, to appear.
2. Gastão de Almeida Braga, The two dimensional lattice Coulomb gas at low temperature, Ph.D. thesis, New York University (1989).
3. J. Dimock and T. R. Hurd, A renormalization group analysis of the Kosterlitz–Thouless phase, preprint (1990).

Communicated by J. L. Lebowitz